for $\frac{d\zeta}{dt}$, which is therefore equal to the constant term of $-\frac{n'^2}{n}\frac{a'^3}{r'^3}$, that is, to $-\frac{n'^2}{n}\left(1+\frac{3}{2}e'^2\right)$. Thus

$$\frac{d\zeta}{dt}=n-\frac{n'^2}{n}\left(1+\frac{3}{2}e'^2_0\right)-\left(\frac{3}{2}\frac{n'^2}{n}-\frac{3675}{64}\frac{n'^4}{n^3}\right)\delta.e'^2.$$

We could have added to the first two terms of this equation a term $B\frac{n'^4}{n^3}e'^2_0$, where B is a numerical coefficient, equal to the aggregate of the constants we have virtually neglected whenever we wrote $\delta\,.\,e'^2$ for e'^2, but it will be easily seen that this would not change the final result. We evidently have

$$n_0=n-\frac{n'^2}{n}\left(1+\frac{3}{2}e'^2_0\right).$$

From which, to a sufficient degree of approximation,

$$n=n_0+\frac{n'^2}{n_0}.$$

Substituting this value of n, we get

$$\frac{d\zeta}{dt}=n_0\left[1+\left(\frac{3}{2}\frac{n'^2}{n_0^2}-\frac{3771}{64}\frac{n'^4}{n_0^4}\right)\left(e'^2_0-e'^2\right)\right].$$

DISCUSSION OF THE GENERAL EQUATION OF THE THIRD DEGREE.

BY JOHN BORDEN, CHICAGO, ILL.

$$x^3+Ax^2+Bx+C=y, \tag{1}$$

$$3x^2+2Ax+B=\frac{dy}{dx}, \tag{2}$$

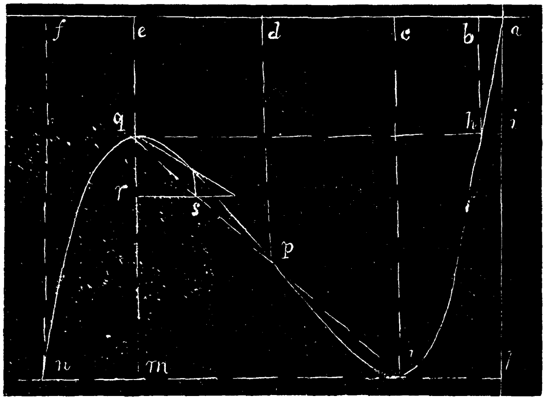
$$6x+2A=\frac{d^2y}{dx^2}. \tag{3}$$

If $\frac{dy}{dx}=0$, then, in (2), $x=-\frac{A}{3}\pm\sqrt{\left(-\frac{B}{3}+\frac{A^2}{9}\right)}$; (4)

and if $\frac{d^2y}{dx^2}=0$, then, in (3), $x=-\frac{A}{3}$. (5)

First Case; A and B positive and $B<\frac{1}{3}A^2$.—The form of the curve is as in the Fig.; its locus being taken for $C=0$.

1. The curve has an inflexion at p; $a\,d = -\frac{1}{3}A$. And from p to the right the curve is concave to the line *cl*, from p to the left it is concave to the line *qm*. Revolve the left hand branch on the point p and it will coincide with the right hand branch, q falling on l.



2. The form of the curve is not dependent on the term C. Move the curve up and down so that the points q and l follow the lines *em* & *cl* and C will vary. For C positive all the roots are negative; and for C negaative, one of the roots is positive. When q coincides with e a case of equal roots occurs, the third root being *ab*. When l coincides with c another case of equal roots occurs, the third root being *af*. When *cl* and *eq* are both negative the only real root refers to the curve from h upward. When *cl* and *eq* are both positive the only real root refers to the curve from n downward.

3. The values of x, in (4), are *ac* and *ae*. Substitute these values in (1) and *cl* and *eq* are found for any assumed value of C. If the results give the same sign there is only one real root, if different signs, there are three real roots. If the values of x in (4) are real, and when substituted in (1) give contrary signs, there are three real roots, otherwise only one real root.

4. To find *ab*, substitute *ae* in (1) and make C such as to reduce y to zero; the third root is *ab*, so, to find *af*, substitute *ac* in (1) and make C such as to reduce y to zero, and the third root is *af*. Or deduct twice *ae* from A and the result is *ab*, so also, deduct twice *ac* from A and the result is *af*.

5. When all the roots are real, one lies between *ab* and *ac*, one between *ac* and *ae* and one between *ae* and *af*.

To find the middle root, suppose C in (1) to have such value that *rs* will coincide with the axis of x, then $ae - rs$ would be the first approximation, and $rs = (ml \times qr) \div qm$, all of which are known, for *qr* is what *eq* becomes in such case. The line *rs*, if above the point of inflexion, is too little as the fiigure shows. The tangent to the curve at the point where the ordinate at s cuts the curve makes an angle with *rs* whose tangent is found by substituting $ae—rs$ for x in (2). The vertical line of the triangle is the value of y in (1) when $ae - rs$ is substituted for x in (1). Hence, by proportion

we reach a formula for the base of the triangle which is the same as Horner's rule. This, if added to *rs*, evidently makes it too much when *rs* is above the point of inflexion. An approximation for the real root between *ac* and *ab*, and for the one between *ae* and *af* can be found by proportion. The curve is distorted in the figure to show the curvature, it in fact approaches more nearly to a straight line.

If C is positive and *cl* and *eq* are both negative, the only real root lies between zero and *ab*, which can be found in the same manner. If C is negative and *cl* and *eq* are both negative, the only real root lies between zero and $—C \div B$. If C is positive and *cl* and *eq* are both positive, then the only real root is greater than *af*.

To find the other limit of this root, substitute *af* for x in (2) and we thus find the tangent of the angle which the tangent to the curve at the point n makes with the axis of abscissas. The curve in such cases cuts the axis of x to the right of the point where this tangent cuts it; and by preportion we find, when the tangent cuts that axis, the vertical of the triangle whose base we seek $= C - qm$.

Secod Case; A negative, B positiv, and $B < \frac{1}{3}A^2$.— In this case substitute $—x$ for x, and we have the previous case.

Third Case; B negative.—In this case the axis of y lies between *cl* and *eq*.

Fourth Case; B positive and $B > \frac{1}{3}A^2$.—In this case the values of x in (2) are imaginary. The points q and l have met at the point of inflexion and the curve from p to the right is concave to the line *dp*, and from p to the left it is concave to the extension of that line below the point p. There can be only one real root, whose locus can be fixed in the manner above indicated.

The higher equations can be treated in the same way. Take the biquadradic

$$x^4+Ax^3+Bx^2+Cx+D=y, \tag{6}$$

$$4x^3+3Ax^2+2Bx+C=\frac{dy}{dx}, \tag{7}$$

$$12x^2+6Ax+2B=\frac{d^2y}{dx^2}. \tag{8}$$

If (7), for $\frac{dy}{dx}=0$, has only one real root, then (6) cannot have more than two real roots. But if $\frac{dy}{dx}=0$ gives three real values for x in (7) which when substituted in (6) gives consecutively contrary signs, then equation (6) has four roots for $y=0$.